Published by Institute of Physics Publishing for SISSA/ISAS

Received: *January 28, 2005*
Accepted: *February 2, 2005*

All-genus calculation of Wilson loops using D-branes

Nadav Drukker

The Niels Bohr Institute, Copenhagen University
Blegdamsvej 17, DK-2100 Copenhagen, Denmark
E-mail: drukker@nbi.dk

Bartomeu Fiol

Institute for Theoretical Physics, University of Amsterdam
1018 XE Amsterdam, The Netherlands
E-mail: bfiol@science.uva.nl

Abstract: The standard prescription for calculating a Wilson loop in the AdS/CFT correspondence is by a string world-sheet ending along the loop at the boundary of AdS. For a multiply wrapped Wilson loop this leads to many coincident strings, which may interact among themselves. In such cases a better description of the system is in terms of a D3-brane carrying electric flux. We find such solutions for the single straight line and the circular loop. The action agrees with the string calculation at small coupling and in addition captures all the higher genus corrections at leading order in α'. The resulting expression is in remarkable agreement with that found from a zero dimensional gaussian matrix model.

Keywords: Strong Coupling Expansion, 1/N Expansion, AdS-CFT and dS-CFT Correspondence, Matrix Models.

Contents

1. Introduction

Some of the most interesting observables in gauge theories are Wilson loop operators, the holonomy of the gauge field around a contour. The expectation value of these operators gives the effective action for a charged particle following that path. A hallmark for confinement is the area-law of the Wilson loop — when its expectation value is proportional to the area enclosed.

In this paper we do not deal with a confining theory, rather with a conformal gauge theory, maximally supersymmetric ($\mathcal{N} = 4$) Yang-Mills, which has a dual description at strong coupling via string theory on $AdS_5 \times S^5$ [1]. Soon after the proposal of this duality the prescription for calculating Wilson loops in string theory was put forth by Rey and Yee and by Maldacena [2, 3]. The prescription is to consider a fundamental string ending on the boundary of AdS along the path specified by the Wilson loop operator.

This suggestion is very intuitive, after all the hope for a string description of confining theories relies on the area of the string bounded by the Wilson loop providing the area law. In the case of the conformal theory the peculiarities of the AdS geometry give an action that is not proportional to the area, but rather is scale independent.

Since its inception, this has been the standard method of evaluating Wilson loop operators in all the generalizations of the AdS/CFT correspondence, including to theories in other dimensions and to confining theories. But here we would like to propose an alternative way of evaluating the Wilson loop operator with D3-branes rather than fundamental strings. One inspiration for this is the concept of "giant gravitons," where fundamental string excitations are replaced by spherical D3-branes wrapping part of the S^5 [4] or part of AdS_5 [5, 6]. Is there an analogous effect for the big strings that describe Wilson loops?

The idea of describing a fundamental string in terms of a D3-brane is actually not new, and to find our solution we follow closely the construction of Callan and Maldacena [7] (see also [8]). There they find a solution to the equations of a D3-brane in flat space in which the brane has a localized spike. This spike is analogous to a string ending on the D3-brane. Here we adapt their calculation to the AdS background and replace the fundamental string describing the Wilson loop with a D3-brane.

The description of the Wilson loop in terms of a fundamental string is a well established part of the AdS/CFT dictionary. So what is the role of D3-brane solutions we find below? The geometry of the branes will be such that they pinch off at the boundary of AdS, ending along the curve defined by the Wilson loop observable. The branes will carry electric flux, which is the same as fundamental string charge, so they can play the same purpose as the fundamental string themselves. In fact, already in one of the original papers on Wilson loops in the AdS/CFT correspondence [2] it was shown that D3 branes can also be used for that purpose.

We will be particularly interested in the case when the Wilson loop is described by a large number of fundamental strings. This happens when the operator involves many coincident Wilson loops, a multiply wound Wilson loop, or a Wilson loop in a high-dimensional representation. Those cases differ by the trace structure in the gauge theory, or the connectivity of the string surfaces [9]. In all three cases the leading planar behavior should be the same and scale like the multiplicity of the loop k. The subleading behavior is an interesting question, that resembles a similar issue in confining theories, as QCD. For confining theories, the flux tube connecting k quarks and k anti-quarks is called a k-string, and its tension σ_k is not just $k\sigma_1$. E.g., for softly broken $\mathcal{N}=2$ SYM, Douglas and Shenker [10] found

$$\sigma_k = N\Lambda^2 \sin\frac{\pi k}{N} = \Lambda^2 \pi k - \frac{\Lambda^2\pi^3}{3!}\frac{k^3}{N^2} + \cdots \tag{1.1}$$

and subsequently this formula has also appeared in MQCD [11] and in supergravity duals of $\mathcal{N}=1$ SYM [12]. We will compare the scaling of both subleading terms in the conclusions.

This subleading behavior may be accounted for by the interaction among the k string surfaces, so it requires to study this system beyond the planar approximation. This is a complicated question and no exact results have ever been calculated. The D3-brane solutions we construct will provide a shortcut to finding all those non-planar contributions!

The reason for this already appeared in [7]. It was argued that the system of many coincident strings is better described in terms of the dynamics of the D3-brane they end on. Applying that logic to the Wilson loop calculation we see that the non-planar contributions, which become important for the multiply wrapped loops should be captured by the D3-brane dynamics.

We will study the two simplest Wilson loop observables, the infinite straight line and the circle. Let us first review the standard calculation of those two Wilson loops. The supersymmetric Wilson loop (in a space of Euclidean signature) is

$$W = \frac{1}{N}\mathrm{Tr}\mathcal{P}\exp i\oint (A_\alpha \dot{x}^\alpha + i\Phi_i|\dot{x}|\theta^i)dt\,, \tag{1.2}$$

where A_α is the gauge field, Φ_i the six scalars, $x^\alpha(t)$ parameterizes a path in space and we take θ^i to be a constant unit vector in $\mathbb{R}^6$.

The simplest Wilson loop imaginable is a single infinite line. Its expectation value corresponds to the exponential of the renormalized mass of the probe particle times the length of the line. In the case of $\mathcal{N}=4$ there should be no renormalization of the mass, and the expected result is simply $\langle W_{\rm line}\rangle = 1$. This can indeed be shown both in the gauge theory and string theory, and is a consequence of this operator preserving half of the supercharges.

To be specific, consider a Wilson loop extended in the x^1 direction and localized in the transverse directions $x^2=x^3=x^4=0$. We use the coordinate system for AdS_5

$$ds^2 = \frac{L^2}{y^2}\left(dy^2+(dx^1)^2+(dx^2)^2+(dx^3)^2+(dx^4)^2\right). \tag{1.3}$$

Here L is the curvature radius of the space, related to the string coupling g_s, string length l_s and the 't Hooft coupling $\lambda = g_{YM}^2 N$ by $L=(4\pi g_s N)^{1/4} l_s = \lambda^{1/4} l_s$.

The standard description of the Wilson loop ending along the line is the surface spanned by the coordinates y and x^1 at $x^2=x^3=x^4=0$. The area of this surface (calculated with the induced metric) will have a divergence from small y, and will be proportional to the length of the x^1 direction, X^1. With a cutoff y_0 the regularized area is

$$A = \sqrt{\lambda}\frac{X^1}{2\pi y_0}\,. \tag{1.4}$$

As explained in [13] (and reviewed below) one should add a boundary term that cancels this divergence. Then the full action is $S_{\rm string}=0$, and indeed the expectation value of the Wilson loop is simply unity.

Next we consider the circular Wilson loop [14, 13], where the path follows a circle of radius R in the x^1, x^2 plane. The string solution describing this loop (using polar coordinates) is given by $r^2+y^2=R^2$ and the bulk action is

$$S_{\rm bulk} = \sqrt{\lambda}\int_{y_0}^R dy\,\frac{r}{y^2}\sqrt{1+r'^2} = \sqrt{\lambda}\int_{y_0}^R dy\,\frac{R}{y^2} = \sqrt{\lambda}\left(\frac{R}{y_0}-1\right). \tag{1.5}$$

The divergence is removed by the boundary term and we are left with the answer

$$\langle W_{\rm circle,string}\rangle = \exp\sqrt{\lambda}\,. \tag{1.6}$$

The final answer is independent of R, due to conformal invariance. The circular Wilson loop is related to the straight line by a conformal transformation, so it also preserves half the supersymmetry [15, 16], yet its expectation value is not unity. For the straight line the combined gluon and scalar propagator vanishes, while for the circle it's a finite constant. This allowed Erickson, Semenoff and Zarembo [17] to sum all rainbow and ladder diagrams and their calculation reproduced the above $\exp\sqrt{\lambda}$ result.

The reason for the difference between the line and the circle is rather subtle. In applying the conformal transformation mapping the line to the circle one needs to add the point at infinity to the line. Consequently there is a slight difference that in perturbation theory manifests itself as an addition of a total derivative to the propagator [18]. Under the assumption that this modification of the propagator is the only change, it was shown there how to write the expectation value of the Wilson loop in terms of zero dimensional gaussian matrix model.

This matrix model was solved exactly and written as an asymptotic expansion to all orders in $1/N$ and $1/\sqrt{\lambda}$. As was mentioned, the leading planar result of the matrix model, $\exp\sqrt{\lambda}$ agrees with the string calculation in AdS. In our case the D3-brane calculation will reproduce this term correctly, as well as an infinite series of corrections of the form $\lambda^{k+1/2}/N^{2k}$. All those terms will be in precise agreement with the perturbative non-planar calculation as given by the matrix model!

The paper is organized as follows. We start with the simple example of the infinite straight line in the next section 2, calculating the action, explaining the necessary boundary terms and proving supersymmetry. In section 3 we turn to the richer case of the circular loop and compare the results to the matrix model. Next we describe how to use D3-branes to calculate 't Hooft loops and comment also on the perturbative calculation of those observables. We end with some discussion.

2. Infinite straight line

As a warm up to explore our idea, we look first at the infinite straight single Wilson line in $\mathbb{R}^4$. It is extended in the x^1 direction and localized in the transverse directions $x^2 = x^3 = x^4 = 0$. This operator exists in both euclidean and lorentzian signature, and our construction will work perfectly well in both. We use Euclidean conventions to be consistent with the other example which we study, the circle. We will switch briefly to lorentzian signature to prove that the solution is supersymmetric. The description of the infinite straight Wilson loop in terms of a D3-brane was originally done by Rey and Yee [2].

2.1 D3-brane solution

To study this Wilson loop it is helpful to use spherical coordinates in the directions transverse to the line, which we choose lo lie along x_1, so for AdS_5 we use the coordinate system

$$ds^2 = \frac{L^2}{y^2}\left(dy^2 + (dx^1)^2 + dr^2 + r^2 d\Omega_2^2\right). \tag{2.1}$$

Here we want to reproduce $\langle W\rangle = 1$, where instead of a fundamental string we use a D3-brane carrying electric flux. This D3-brane will be a hypersurface in AdS_5 given by a single equation, which by the symmetries of the problem is clearly $y = y(r)$. So we use x^1, r, θ and ϕ as world-volume coordinates and turn on the scalar field $y(r)$ as well as the electric field $F_{1r}(r)$.

The action includes the Dirac-Born-Infeld (DBI) part and the Wess-Zumino (WZ) term, which captures the coupling to the background Ramond-Ramond fields. In AdS_5 there are N units of flux of the Ramond-Ramond five-form, whose potential can be taken to be

$$C_4 = \frac{L^4 r^2 \sin\theta}{y^4} dx^1 \wedge dr \wedge d\theta \wedge d\phi \,. \tag{2.2}$$

We thus find the action

$$\begin{aligned} S &= T_{\rm D3} \int e^{-\Phi} \sqrt{\det(g + 2\pi\alpha' F)} - T_{\rm D3} \int P[C_4] \\ &= \frac{2N}{\pi} \int dx^1 dr \, \frac{r^2}{y^4} \left(\sqrt{1 + y'^2 + (2\pi\alpha' F_{1r})^2 \frac{y^4}{L^4}} - 1 \right) . \end{aligned} \tag{2.3}$$

Here we used a prime to denote derivation by r, and $P[C_4]$ is the pullback of the four-form to the world-volume. The tension of the D3-brane, $T_{\rm D3}$, is given by

$$T_{\rm D3} = \frac{1}{(2\pi)^3 l_s^4 g_s} = \frac{N}{2\pi^2 L^4} \,. \tag{2.4}$$

Since the world-volume gauge field A_1 does not appear explicitly in the action, its conjugate momentum $i\Pi$ is conserved (in the Euclidean theory the electric field is imaginary, so with this extra i we will get a real quantity). It is

$$\Pi = -i \frac{4N}{\lambda} \frac{2\pi F_{1r} r^2}{\sqrt{1 + y'^2 + 4\pi^2 F_{1r}^2 y^4/\lambda}} \,. \tag{2.5}$$

In this definition we integrated the momentum over the S^2, which gave the conserved charge corresponding to the fundamental string density. So Π will be an integer, k, which corresponds to the number of coincident Wilson loops.

Motivated by the spike solution in flat space [7],[1], we consider the linear ansatz $y = r/\kappa$. Plugging this into the equations of motion we find that it solves them for the constant $\kappa = k\sqrt{\lambda}/4N$. This gives the electric field

$$F_{1r} = i \frac{k\lambda}{8\pi N r^2} \,. \tag{2.6}$$

This solution is in fact a limit of one described in [19], where they consider a D3-brane probe in the background generated by other D3-branes. Ours is just the near horizon limit of their solution.

[1] In flat space the transverse coordinate behaved like $X^9 \sim 1/r$. This coordinate X^9 is replaced here by $1/y$, hence the linear ansatz.

As stated, the solution is a hypersurface in AdS_5, and the induced metric on the brane is given by

$$ds^2 = \frac{L^2\kappa^2}{r^2}\left((1+\kappa^{-2})dr^2 + dt^2\right) + L^2\kappa^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{2.7}$$

Thus the geometry has the product structure[2] $AdS_2 \times S^2$. The curvature radius of the AdS_2 factor is $L\sqrt{1+\kappa^2}$ and of the S^2 is $L\kappa$. Having this product structure means that the sphere never shrinks, even as we approach the boundary of AdS_5. But in the dual CFT it corresponds to a point, and not to a finite size sphere due to the infinite rescaling of the metric near the boundary of AdS [22].

Calculating the action for this solution we find that the WZ term exactly cancels the DBI part, giving $S = 0$. This is the expected final answer, but the calculation is not complete. Thus far we only considered the bulk action, one should add to this appropriate boundary terms, to which we will turn now.

2.2 Boundary terms

The D3-brane solution we found extends all the way to the boundary of AdS_5 and ends there along a one-dimensional curve. This opens up the possibility of adding boundary terms to the action. These boundary terms don't change the equations of motion, so the solution is still the same, but the value of the action when evaluated at this solution will in general depend on the boundary terms.

When calculating the Wilson loop using string surfaces the bulk action is divergent, but this term is fixed by a boundary term [13]. Let us recall the argument, since we will have to apply it for the case at hand.

The string used to describe the Wilson loop has to satisfy complementary boundary conditions with respect to a free string ending on a D3-brane. While the latter has to satisfy Dirichlet conditions for six directions and Neumann conditions for the other four, the string describing the Wilson loop has to satisfy six Neumann conditions and four Dirichlet. An easy way to convince oneself of this fact is to consider a Wilson loop on a D9-brane, which has to follow a curve in ten-dimensions, hence ten Dirichlet conditions, after six T-dualities one gets the D3-brane and the boundary conditions stated above.

The Dirichlet boundary conditions are on the four directions parallel to the boundary of AdS, and the six Neumann ones combine the radial coordinate of AdS and the S^5 coordinates. The Nambu-Goto action for the string (as well as the Polyakov action) is a functional of the coordinates, which is the appropriate action assuming we have Dirichlet boundary conditions. So we have to add boundary terms that change the boundary conditions.

Since all the Wilson loops we discuss have no dependence on the S^5, the only coordinate we have to replace with its momentum is the radial coordinate y. We therefore define p_y

[2] D-brane solutions with $AdS \times S$ induced geometry have appeared before in the literature, see e.g. [20, 21]; an important difference is that in our case the entire brane, including the S^2, is embedded in AdS_5, while in those examples the sphere part is inside the sphere of target space.

as the momentum conjugate to it

$$p_y = \frac{\delta S}{\delta \partial_n y}\,, \tag{2.8}$$

where ∂_n is the normal derivative to the boundary.

The new action including the term that changes the boundary conditions is

$$\tilde{S} = S - y_0 \int d\tau\, p_y\,, \tag{2.9}$$

where the integral is over the boundary at a cutoff $y = y_0$. The original action S is a functional of y and ∂y. Applying the standard variational technique to it we find

$$\delta S = \int d^2\sigma \left[\frac{\delta S}{\delta y}\delta y + \frac{\delta S}{\delta \partial y}\delta\partial y\right] = \oint d\tau\, p_y \delta y\,, \tag{2.10}$$

where the bulk part part vanishes due to the equations of motion. The boundary term clearly indicates that it is a functional of y. Including the boundary term, the variation of the new action is

$$\delta\tilde{S} = -\oint d\tau\, y \delta p_y\,, \tag{2.11}$$

and it's indeed a functional of p_y, as advertised.

When calculating the action for the fundamental string this boundary term cancels the divergence in the area. In our calculation, using D3-branes the action was finite (zero), so we do not need to cancel a divergence, but the logic that applied to the string still holds, and we should apply the same procedure here. The DBI action is a functional of the coordinates, and in particular of y and y', so we have to add the same kind of boundary term as for the string.

Using our action (2.3) we find that the momentum conjugate to y (integrated over the sphere) is

$$p_y = \frac{2N}{\pi} \frac{r^2 y'}{y^4\sqrt{1 + y'^2 + (2\pi\alpha' F_{1r})^2 \frac{y^4}{L^4}}}\,. \tag{2.12}$$

Using the equations of motion we get the boundary term inserted at a cutoff y_0

$$-\int dx^1\, y_0 p_y = -\frac{2N}{\pi}\frac{X^1 \kappa}{y_0}\,. \tag{2.13}$$

Note that for the bulk part of the DBI action diverged like κ^3/y_0, so for small κ the boundary term is much larger than the bulk contribution. It is actually the same as for the fundamental string. There it exactly canceled the bulk term, since the world-sheet was perpendicular to the boundary, i.e. in the y direction. The D3-brane extends also in the r direction and has non-zero momentum in that direction, that is why p_y is not equal to the bulk action.

Since we still expect the full action to vanish, there must be another boundary term, which is the Legendre transform of the other variable, the gauge field. The action (2.3) is a functional of the gauge field, but the Wilson loop observable defines the number k, which is the dimension of the representation of the loop, or the wrapping number, for a multiply wound loop. The momentum conjugate to the the gauge field, Π, calculated in (2.5) is precisely equal to k, therefore it's the correct variable to use, instead of the gauge field. Thus we find another boundary contribution to the action, which we write as the integral over the total derivative

$$-\int dx^1\, i\Pi A_1 = -\int dx^1\, dr\, i\Pi F_{1r} = \frac{2N}{\pi}\frac{X^1\kappa}{y_0}\,. \tag{2.14}$$

The sum of the two Legendre transforms — of the coordinate y and of the gauge field — add up to zero. So on-shell there is no boundary contribution to the action and we are left with a total action $S=0$, or $\langle W\rangle = 1$, as expected from supersymmety.

There is another subtlety associated with the boundary. The Wess-Zumino part is not well defined on a manifold with boundary. It should be the integral of the five-form flux surrounded by the brane

$$S_{\mathrm{WZ}} = -T_{\mathrm{D3}}\int_{M_5} F_5\,. \tag{2.15}$$

For a D3-brane without boundaries this is well defined (up to replacing the inside and outside), but it's not clear what M_5 should be for a D3-brane with boundary.

We defined the action in terms of the pullback of the 4-form potential

$$S_{\mathrm{WZ}} = -T_{\mathrm{D3}}\int_{M_4} P[C_4]\,. \tag{2.16}$$

Under gauge transformations $\delta C_4 = d\Lambda$ (since the NS 2-form vanishes). So the variation of the action is

$$\delta S_{\mathrm{WZ}} = -T_{\mathrm{D3}}\int_{M_4} P[\delta C_4] = -T_{\mathrm{D3}}\int_{\partial M_4} P[\Lambda]\,. \tag{2.17}$$

We used a very natural form of C_4 (2.2) possessing the relevant symmetries of our problem and no singularities along the Wilson loop. Other choices may lead to different answers (see the discussion in section 3.5). One may fix this ambiguity by hand — adding a three-form on the boundary (equal to zero in our gauge) and imposing that it transforms by $P[\Lambda]$ under gauge transformations. It would be good to get a better understanding of this issue.

2.3 Supersymmetry

The infinite straight Wilson loop preserves half of the supersymmetries of the theory, as does the string solution in AdS. Here we will show that the D3-brane preserves the same supersymmetries.

In order to check supersymmetry we switch to lorentzian signature and define the vielbeins

$$e_y^{\bar{y}} = \frac{L}{y}\,,\qquad e_t^{\bar{t}} = \frac{L}{y}\,,\qquad e_r^{\bar{r}} = \frac{L}{y}\,,\qquad e_\theta^{\bar{\theta}} = \frac{Lr}{y}\,,\qquad e_\phi^{\bar{\phi}} = \frac{Lr\sin\theta}{y}\,. \tag{2.18}$$

We use Γ_a as constant gamma matrices and define $\gamma_\mu = e_\mu^a\Gamma_a$. Using two constant spinors of positive and negative chirality $\epsilon_0^\pm$ that satisfy also $i\Gamma_{\bar{t}\bar{r}\bar{\theta}\bar{\phi}}\epsilon_0^\pm = \pm\epsilon_0^\pm$. and the matrix

$$M = \exp\left(\frac{\theta}{2}\Gamma_{\bar{r}\bar{\theta}}\right)\exp\left(\frac{\phi}{2}\Gamma_{\bar{\theta}\bar{\phi}}\right), \tag{2.19}$$

the Killing spinors of AdS_5 are written as[3] (see for example [21])

$$\epsilon = y^{-1/2} M \epsilon_0^- + \left(y^{1/2} \Gamma_{\bar{y}} + y^{-1/2} (r \Gamma_{\bar{r}} + t \Gamma_{\bar{t}}) \right) M \epsilon_0^+ . \tag{2.20}$$

They satisfy the equation $D_\mu \epsilon = \frac{i}{2L} \Gamma_{\bar{t}\bar{r}\bar{\theta}\bar{\phi}\bar{y}} \gamma_\mu \epsilon$.

The supersymmetries preserved by the D3-brane are generated by the Killing spinors that also satisfy $\Gamma\epsilon = \epsilon$ where Γ is the projector associated with the D3-brane. In our case it is given by

$$\Gamma = \frac{1}{\sqrt{-\det(g + 2\pi\alpha' F)}} \left[(y' \gamma_{ty\theta\phi} + \gamma_{tr\theta\phi}) I - 2\pi\alpha' F_{tr} \gamma_{\theta\phi} K I \right] , \tag{2.21}$$

with K acts on spinors by complex conjugation and I multiplies them by $-i$. For our solution the square root in the denominator is equal to $L^4 r^2 \sin\theta / y^4$ while $y' = 1/\kappa$ and in the lorentzian theory $2\pi\alpha' F_{tr} = L^2 \kappa / r^2$.

Naively it would seem like the equation $\Gamma\epsilon = \epsilon$ would impose two conditions on the spinors, so only 1/4 of the supersymmetries would be preserved. This is in fact what happens in the flat space case of Callan and Maldacena [7]. But since AdS is the background created by D3-branes, the projector associated with the D3-brane does not break any of the supersymmetries.

To see that we rewrite Γ as

$$\Gamma = \left[1 - \kappa^{-1} \Gamma_{\bar{r}} \left(\Gamma_{\bar{y}} - \Gamma_{\bar{t}} K \right) \right] \Gamma_{\bar{t}\bar{r}\bar{\theta}\bar{\phi}} I , \tag{2.22}$$

and the simplification arises since $\epsilon_0^\pm$ are eigenstates of $\Gamma_{\bar{t}\bar{r}\bar{\theta}\bar{\phi}} I$. A bit of algebra gives

$$\Gamma\epsilon - \epsilon = -\kappa^{-1} y^{-1/2} \Gamma_{\bar{r}} M (\Gamma_{\bar{y}} - \Gamma_{\bar{t}} K) \epsilon_0^- + \kappa^{-1} y^{-1/2} \Gamma_{\bar{r}} \left(t \Gamma_{\bar{t}} - r \Gamma_{\bar{r}} - y \Gamma_{\bar{y}} \right) M \left(\Gamma_{\bar{y}} + \Gamma_{\bar{t}} K \right) \epsilon_0^+ . \tag{2.23}$$

Thus the system will be invariant under supersymmetries generated by ϵ made up of $\epsilon_0^\pm$ subject to the constraints

$$\Gamma_{\bar{y}} \epsilon_0^- = \Gamma_{\bar{t}} \epsilon_0^{-*} \qquad \Gamma_{\bar{y}} \epsilon_0^+ = -\Gamma_{\bar{t}} \epsilon_0^{+*} , \tag{2.24}$$

where $\epsilon_0^{\pm *}$ is the complex conjugate of $\epsilon_0^\pm$.

3. Circular loop

After proving the feasibility of using a D3-brane to calculate a Wilson loop in the case of the straight line, we turn now to the more interesting case of the circular Wilson loop.

[3]Since our solution has no dependence on the S^5 part of the geometry, we do not need the form of the Killing spinors on the full $AdS_5 \times S^5$. To account for that, one simply has to multiply the constant spinors $\epsilon_0^\pm$ with a function of the S^5 coordinates and gamma matrices.

JHEP02(2005)010

3.1 Bulk calculation

Let's start with the coordinate system for AdS_5

$$ds^2 = \frac{L^2}{y^2}\left(dy^2 + dr_1^2 + r_1^2 d\psi^2 + dr_2^2 + r_2^2 d\phi^2\right), \tag{3.1}$$

where r_1 is the radial coordinate in the x^1, x^2 plane and r_2 is the radial coordinate in the x^3, x^4 plane. We place the Wilson loop at $r_1 = R$, and $r_2 = 0$. We want to find a D3-brane solution of the DBI action, pinching to this circle as $y \to 0$. To find the solution, it turns out to be more convenient to change to the coordinates ρ, θ, and η defined by[4]

$$r_1 = \frac{R\cos\eta}{\cosh\rho - \sinh\rho\cos\theta}, \quad r_2 = \frac{R\sinh\rho\sin\theta}{\cosh\rho - \sinh\rho\cos\theta}, \quad y = \frac{R\sin\eta}{\cosh\rho - \sinh\rho\cos\theta}. \tag{3.2}$$

With these coordinates the metric of AdS_5 is given by

$$ds^2 = \frac{L^2}{\sin^2\eta}\left(d\eta^2 + \cos^2\eta d\psi^2 + d\rho^2 + \sinh^2\rho\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right). \tag{3.3}$$

These coordinates cover the space once if they take the ranges $\rho \in [0, \infty)$, $\theta \in [0, \pi]$ and $\eta \in [0, \pi/2]$. The boundary of space $y = 0$ is mapped to $\eta = 0$ as well as $\rho \to \infty$. The circle on the boundary is located at $\eta = \rho = 0$.

The string solution describing the circular Wilson loop is given by $r_1^2 + y^2 = R^2$, or in the new coordinate system by $\rho = 0$. The bulk action is then

$$S_{\text{bulk}} = \frac{\sqrt{\lambda}}{2\pi}\int d\eta\, d\psi\, \frac{\cos\eta}{\sin^2\eta} = \sqrt{\lambda}\left(\frac{1}{\sin\eta_0} - 1\right), \tag{3.4}$$

with η_0 a cutoff on η. The divergence is removed by the boundary term and we are left with the answer

$$\langle W_{\text{circle,string}}\rangle = \exp\sqrt{\lambda}\,. \tag{3.5}$$

We wish now to find the appropriate D3-brane that ends along the circle. Again it will be given by a single equation, and the symmetries guarantee that it is of the form $\eta = \eta(\rho)$. So we may take ψ, ρ, θ and ϕ as the world-volume coordinates. There is a single scalar field η and a single component of the electromagnetic field $F_{\psi\rho}(\rho)$. We take the four-form potential, C_4, to be the same as for the straight line. In the (r_1, r_2) coordinates this is just

$$C_4 = L^4\frac{r_1 r_2}{y^4} dr_1 \wedge d\psi \wedge dr_2 \wedge d\phi\,. \tag{3.6}$$

and in the new coordinates it is

$$\begin{aligned} C_4 = {} & L^4\frac{\cos^2\eta\sin\theta\sinh^2\rho}{\sin^4\eta} d\rho\wedge d\psi\wedge d\theta\wedge d\phi + \\ & + L^4\frac{\cos\eta\sin\theta\sinh^2\rho(\sinh\rho - \cosh\rho\cos\theta)}{\sin^3\eta(\cosh\rho - \sinh\rho\cos\theta)} d\eta\wedge d\psi\wedge d\theta\wedge d\phi - \\ & - L^4\frac{\cos\eta\sin^2\theta\sinh\rho}{\sin^3\eta(\cosh\rho - \sinh\rho\cos\theta)} d\eta\wedge d\psi\wedge d\rho\wedge d\phi\,. \end{aligned} \tag{3.7}$$

[4]These coordinates have some resemblance to the ones used to describe black rings, see e.g. [23].

Using this the DBI part of the action is

$$\begin{aligned} S_{\rm DBI} &= T_{\rm D3}\int e^{-\Phi}\sqrt{\det(g+2\pi\alpha' F)} \\ &= 2N\int d\rho\, d\theta\, \frac{\sin\theta\sinh^2\rho}{\sin^4\eta}\sqrt{\cos^2\eta(1+\eta'^2)+(2\pi\alpha')^2\frac{\sin^4\eta}{L^4}F_{\psi\rho}^2}\,. \end{aligned} \tag{3.8}$$

The WZ part is

$$\begin{aligned} S_{\rm WZ} &= -T_{\rm D3}\int P[C_4] \\ &= -2N\int d\rho\, d\theta\, \frac{\cos\eta\sin\theta\sinh^2\rho}{\sin^4\eta}\left(\cos\eta+\eta'\sin\eta\,\frac{\sinh\rho-\cosh\rho\cos\theta}{\cosh\rho-\sinh\rho\cos\theta}\right). \end{aligned} \tag{3.9}$$

Again Π, the momentum conjugate to the gauge field, is conserved, and is equal to the fundamental string charge, or the number of coincident Wilson loops. Now the ansatz $\sin\eta=\kappa^{-1}\sinh\rho$ solves the equations of motion if[5]

$$k=\Pi=\frac{4N\kappa}{\sqrt{\lambda}}\,,\qquad \text{which gives}\qquad F_{\psi\rho}=\frac{ik\lambda}{8\pi N\sinh^2\rho}\,. \tag{3.10}$$

It should be now obvious why we chose this coordinate system. For one it preserves the symmetry of the problem, as there is no θ dependence. But more than that, near the boundary, where η is small we see that the linear approximation to the solution is the same as for the straight line, with the replacements $\rho\to r$ and $\eta\to y$. This clearly will always be true, since in the UV all smooth loops look like the straight line.

The induced metric on the D3-brane is given by

$$ds^2=\frac{L^2}{\sin^2\eta}\left(\frac{1+\kappa^2}{1+\kappa^2\sin^2\eta}d\eta^2+\cos^2\eta d\psi^2\right)+L^2\kappa^2\left(d\theta^2+\sin^2\theta d\phi^2\right), \tag{3.11}$$

which again is the metric of $AdS_2\times S^2$ (to see that one can switch to the coordinate ζ defined by $\cot^2\eta=(1+\kappa^2)\sinh^2\zeta$). As before the radius of the AdS_2 factor is $L\sqrt{1+\kappa^2}$ and of the sphere $L\kappa$. The main difference is that here we find the global structure of AdS_2 is the Poincaré disc, while before it was the upper half plane. This is the same kind of difference found between the string solutions describing those two Wilson loops. The D3-brane hypersurface is depicted in figure 1. in the r_1, r_2 and y coordinate system.

Like the straight line, this solution also preserves half the supersymmetries.

Both the DBI and WZ parts of the action suffer from a divergence near the boundary, but plugging the solution into the action we find the combination to be perfectly finite

$$\begin{aligned} S_{\rm DBI+WZ} &= 2N\kappa^2\int d\rho\, d\theta\, \frac{\sin\theta\cos\theta}{\sinh\rho(\cosh\rho-\sinh\rho\cos\theta)} \\ &= 2N\kappa^2\left[\coth\rho-\frac{\rho}{\sinh^2\rho}\right]_{\rho=0}^{\sinh\rho=\kappa}\,. \end{aligned} \tag{3.12}$$

There is no contribution from the lower limit of the integral, near the boundary. So the bulk action is the above expression evaluated at $\sinh\rho=\kappa$. Before studying it further we turn to the boundary terms.

[5] Again we defined Π as $-i$ times the conjugate to F, to make it real. Also it is defined integrated over θ and ϕ, but not over ψ, so it corresponds to the effective fundamental string density.

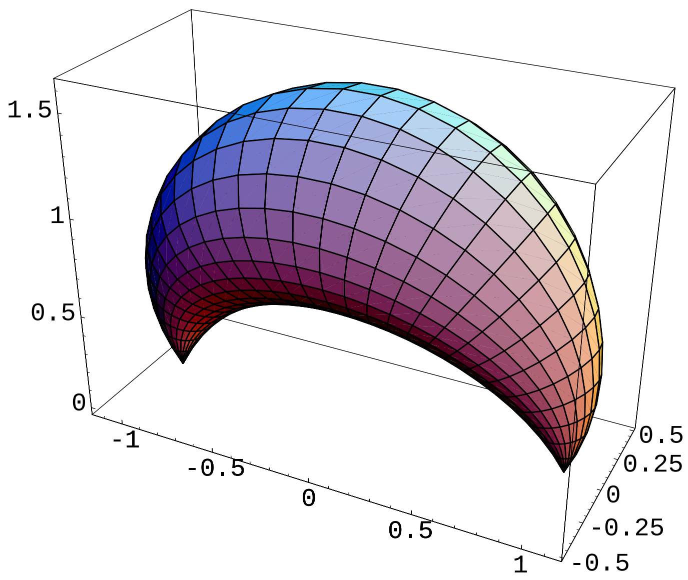

Figure 1: A depiction of the D3-brane solution describing the circular Wilson loop of radius $R = 1$. The horizontal plane in the figure is the r_1, r_2 plane and the vertical direction is y, the two angular directions ψ and ϕ are suppressed. The surface pinches off at the boundary of AdS (the bottom of the picture) at $r_2 = 0$ and $r_1 = 1$ (also $r_1 = -1$), and expands away from it.

3.2 Boundary terms

As in the case of the straight line we have to complement the bulk calculation with boundary terms. The first of these is the Legendre transform of the radial coordinate in AdS. In our coordinate system (3.3) the boundary is given by $\eta \to 0$ (also $\rho \to \infty$, but that regime is far from our D3-brane). We are more used to the radial coordinate y of (3.1), but the two are proportional to each other near the boundary, up to $O(y^3)$ corrections. So the prescription for the Wilson loop involves the Legendre transform of η.

The momentum conjugate to η is[6]

$$p_\eta = \frac{\delta L}{\delta \eta'} = \frac{N}{2\pi^2} \frac{\sin\theta \sinh^2\rho \cos^2\eta}{\sin^4\eta} \left[\eta' - \tan\eta \, \frac{\sinh\rho - \cosh\rho\cos\theta}{\cosh\rho - \sinh\rho\cos\theta}\right], \tag{3.13}$$

where the first term comes from the DBI part and the second from the WZ piece. The resulting boundary term is

$$-\eta_0 \int p_\eta = -4N\frac{\kappa}{\eta_0} + O(\eta_0)\,. \tag{3.14}$$

Here the WZ part contributed only terms that vanish in the $\eta_0 \to 0$ limit, and the result is the same as for the straight line (except that the length of the line X^1 is replaced by 2π).

Next we perform the Legendre transform on the gauge field, replacing it with the conjugate momentum $\Pi = k$. As in the straight line case we do that by adding the total

[6]Here it is defined as not integrated over the S^2, since it has nontrivial θ dependence.

derivative

$$-\int d\rho\, d\psi\, i\Pi F_{\psi\rho} = -4N\kappa^2 \coth\rho \Big|_{\sinh\rho=\kappa\sin\eta_0}^{\sinh\rho=\kappa} . \tag{3.15}$$

The contribution from the lower limit is equal to $4N\kappa/\eta_0$ and exactly cancels the divergence from the other boundary term. The contribution from the upper limit combines with the bulk term (3.12) to give

$$S_{\text{circle}} = -2N\left[\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa\right]. \tag{3.16}$$

As before there should also be a boundary term to make the Wess-Zumino part of the action gauge invariant. Since we did not find a compelling expression for this term, we leave it open.

3.3 Analysis of the result

The resulting expression for the action of the D3-brane (3.16) is rather complicated, let us expand it in the regime we are most familiar with, where $\lambda \ll N^2$ and k is small, or small κ.

This gives

$$S_{\text{circle}} = -4N\kappa - \frac{2N\kappa^3}{3} + \frac{N\kappa^5}{10} + O(\kappa^7) = -k\sqrt{\lambda} - \frac{k^3\lambda^{3/2}}{96N^2} + \frac{k^5\lambda^{5/2}}{10240N^4} + O\left(\frac{\lambda^{7/2}}{N^6}\right). \tag{3.17}$$

The first term in the final expression is the same as the action of k coincident strings (1.5), but the full result includes an infinite series of corrections in $1/N^2$. Can those terms be explained in terms of the fundamental strings?

The explanation to these terms was given in fact in [18], so let us review it. We want to examine the string loop contributions to the Wilson loop as calculated using fundamental strings in AdS. At large λ we would be instructed to find classical minimal surfaces of higher genus ending on the curve on the boundary. Such smooth solutions will not exist, instead we should consider the original solution with degenerate handles attached to it.

So at order g_s^{2p} we should consider p indistinguishable degenerate handles ending on our surface. They will have the same action as the leading planar result, but with a different prefactor. The string coupling gives the obvious factor $g_s^{2p} \sim (\lambda/N)^{2p}$. In addition we should worry about the measure of integration. A generic open string with one boundary and p handles will have $6p-3$ real moduli. In the large λ limit we have to consider only degenerate handles, which imposes two constraints per handle (so each handle is left with four real moduli, the locations of its two ends). Each of those constraints (in addition to the overall three) introduce a delta function into the integration measure that will give a factor of the inverse effective length-scale of the problem, i.e. $\lambda^{-1/4}$. Together with the combinatorial factor we expect therefore at order $2p$ a result proportional to

$$\frac{1}{p!}\frac{g_s^2}{\lambda^{(2p+3)/4}} \sim \frac{1}{p!}\frac{\lambda^{(6p-3)/4}}{N^{2p}} . \tag{3.18}$$

Those corrections will all exponentiate to give a term proportional to $\lambda^{3/2}/N^2$ in the expectation value of the Wilson loop. If there are k coincident string surfaces our calculation shows that the result will scale with k^3, but we don't have a good heuristic argument for this scaling.

Getting this term assumed that the handles are all independent. There will be also contributions when two (or more) of those handles collide, which are even more degenerate surfaces. Those surfaces will have higher genus, and smaller measure, resulting in terms like $\lambda^{5/2}/N^4$ and so on.

It is amusing to look separately at the bulk and boundary contributions. The bulk contribution alone gives

$$S_{\text{circle,bulk}} = \frac{k^3\lambda^{3/2}}{48N^2} + O\left(\frac{k^5\lambda^{5/2}}{N^4}\right), \tag{3.19}$$

so it does not include the term linear in k. To get those correctly it was crucial to include the boundary terms that made the action a functional of the correct variables, the momenta p_η and Π.

We have proposed a different prescription for calculating Wilson loops, by using D3-branes rather than fundamental strings. Are both calculations equally good, or is there a reason to prefer one over the other?

This issue was addressed in the flat-space case [7] and two criteria were found for the validity of the D-brane calculation. The first requirement was that the fields on the brane will be slowly varying, and the other is that the system does not back-react on the geometry.

Looking at the D3-brane world-volume, it has the product structure $AdS_2 \times S^2$. The radius of curvature of the AdS_2 factor is $L\sqrt{1+\kappa^2}$, so it never becomes small. The radius of curvature of the S^2 part is $L\kappa$. The calculation cannot be trusted unless this radius is larger than l_s, which translates into requiring $\kappa \gg \lambda^{-1/4}$. To prevent the system to back-react on the geometry we have to impose $kg_s^2 \ll 1$. In terms of our variables this translates to $\kappa \ll 1/(g_s\sqrt{\lambda})$. Note that since we always assume $\lambda \gg 1$, the range of validity includes the regime of small κ, where the first term, $-4N\kappa = -k\sqrt{\lambda}$ dominates.

In some ways the D3-brane solution in AdS is better than in flat space [7]. In flat space the D3-brane world-volume gets highly curved near the source of the electric field, and the field strength and its derivatives diverge. The induced metric on our solution is homogeneous, so our requirement will lead to small curvature on the entire brane, not only in some asymptotic part.

The specific cases studied in this paper are supersymmetric, and the calculation seems to work beyond the expected range of validity. As we will see in the next subsection, the result we obtained matches with a matrix model computation, assuming just $\lambda \gg 1$ with no other restrictions on κ.

Let us comment about the magnetic case (studied below). The requirement of no back-reaction is now $k\tilde{g}_s \ll 1$, which we wrote in terms of the dual couplings $\tilde{g}_s = 1/g_s$. In the magnetic case κ is defined as $\kappa = \pi k/\sqrt{\tilde{\lambda}}$ (where $\tilde{\lambda} = 16\pi^2 N^2/\lambda$) which leads to the same requirement as in the electric case $\kappa \ll 1/(g_s\sqrt{\lambda})$. The condition on the radius of the sphere will again be $\kappa \gg \tilde{\lambda}^{-1/4}$, so with those definitions of κ we find the same range of validity for the electric and magnetic cases.

3.4 Comparison with the matrix model

As already mentioned before, the circular Wilson loop has very nice properties when calculated in perturbation theory. The combined propagator of the gauge field and scalar is a constant, which reduces the calculation of all rainbow/ladder diagrams to a zero-dimensional matrix model [17], which can be written explicitly as an integral over all $N \times N$ hermitean matrices M

$$\langle W_{\text{ladders}}\rangle = \left\langle \frac{1}{N}\mathrm{Tr}\exp M \right\rangle = \frac{1}{Z}\int \mathcal{D}M \frac{1}{N}\mathrm{Tr}e^{M} e^{-\frac{2N}{\lambda}\mathrm{Tr}M^2}\,. \tag{3.20}$$

The leading behavior at large N, expressed in terms of the Bessel function I_1, is easily found using Wigner's semi-circle law and is

$$\langle W_{\text{ladders}}\rangle \sim \int_{-1}^{1} dx\,\sqrt{1-x^2}\,e^{x\sqrt{\lambda}} = \frac{2}{\sqrt{\lambda}} I_1\left(\sqrt{\lambda}\right) \sim e^{\sqrt{\lambda}}\,, \tag{3.21}$$

which is indeed the leading behavior of the circular Wilson loop as calculated by a string in AdS.

One can do better and solve this matrix model exactly applying several different techniques. In [18] this was done using orthogonal polynomials to give the full result at finite N, then the result was rearranged in a $1/N$ expansion. Using those expressions the leading $\sqrt{\lambda}$ term was reproduced as well as an infinite series of corrections. It was noticed there that some of the terms in this series exponentiated to give $\exp[\lambda^{3/2}/(96N^2)]$, exactly as in (3.17). From equation (B.7) of that paper the next term, $\lambda^{5/2}/(10240N^4)$, can also be extracted with the correct coefficient.

To check our result more closely let us recall the result for the matrix model at finite N. It was given in terms of a Laguerre polynomial[7] as

$$\langle W_{\text{ladders}}\rangle = \left\langle \frac{1}{N}\mathrm{Tr}\exp M \right\rangle = \frac{1}{N} L^1_{N-1}(-4N\kappa^2)\exp[2N\kappa^2]\,, \tag{3.22}$$

The multiply wound Wilson loop is given by the expectation value of the matrix model operator $\mathrm{Tr}\exp kM$, which amounts to the replacement $\lambda \to k^2\lambda$. That allowed us to express λ in the matrix model result in terms of N and κ.

One could consider other operators, like $(\mathrm{Tr}\exp M)^2$, which would correspond to two overlapping Wilson loops. While the planar result will always scale with the total multiplicity $\exp k\sqrt{\lambda}$, the exact expressions are more complicated [18], and one should not expect the same answer as the multiply wrapped loop beyond the planar limit.

Going back to the case of the single trace operator, it is expressed in terms of a Laguerre polynomial which satisfies the differential equation

$$xL^k_n(x)'' + (k+1-x)L^k_n(x)' + nL^k_n(x) = 0\,. \tag{3.23}$$

[7] $L^k_n(x) = 1/n!\exp[x]x^{-k}(d/dx)^n(\exp[-x]x^{n+k})$

This leads to the following differential equation for the expectation value of the Wilson loop

$$\left[\kappa\partial_\kappa^2 + 3\partial_\kappa - 16N^2\kappa(1+\kappa^2)\right]\langle W_{\text{ladders}}\rangle = 0\,. \tag{3.24}$$

Next we rewrite our observable as the exponent of an effective action $\langle W_{\text{ladders}}\rangle = \exp[-N\,\mathcal{F}]$ and derive the equation for $\mathcal{F}(\kappa)$

$$(\mathcal{F}')^2 - \frac{1}{N\kappa}\left(\kappa\mathcal{F}'' + 3\mathcal{F}'\right) - 16(1+\kappa^2) = 0\,. \tag{3.25}$$

Since $N\kappa \sim k\sqrt{\lambda}$ and we are in the regime where $\lambda \gg 1$, we will neglect the terms subleading in N and $N\kappa$, to find

$$\frac{d\mathcal{F}}{d\kappa} = \pm 4\sqrt{1+\kappa^2}\,. \tag{3.26}$$

Finally we integrate this to find

$$\mathcal{F} = \mathcal{F}_0 \pm 2\left[\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa\right]. \tag{3.27}$$

Fixing the boundary condition $\mathcal{F}_0 = 0$ and the sign $-$ from the explicit results of the matrix model stated above, we get full agreement with the D3-brane calculation (3.16)

$$S = N\mathcal{F}\,. \tag{3.28}$$

Notice that to match both computations, we only had to require $N\kappa \gg 1$, or equivalently $\lambda \gg 1$. So the match works even for $k = 1$.

3.5 Conformal transformation

While we presented the calculation of the circular Wilson loop as independent from the straight line, the two are actually intimately connected. After all, the straight line and the circle are related by a conformal transformation, and that conformal transformations in the boundary of AdS extend to isometries of the full AdS space. Therefore, one may obtain the D3-worldvolume associated to the circular Wilson loop by the (extension of the) conformal transformation that takes the line to the circle[8]. It is illuminating to carry out this exercise.

The special conformal transformations generated by a vector c^α on the boundary is extended to the isometry of AdS

$$\begin{aligned} x^\alpha &= \frac{\tilde{x}^\alpha + c^\alpha((\tilde{x})^2 + \tilde{y}^2)}{1 + 2c\cdot\tilde{x} + (c)^2((\tilde{x})^2 + \tilde{y}^2)}\,, \\ y &= \frac{\tilde{y}}{1 + 2c\cdot\tilde{x} + (c)^2((\tilde{x})^2 + \tilde{y}^2)}\,. \end{aligned} \tag{3.29}$$

The inverse is given by the same equations with $c^\alpha \to -c^\alpha$.

Starting with the straight line we first have to shift it away from the origin in the x^2 direction to $(x^1, 1/2, 0, 0)$, then using the above transformation (with $c = (0, -1, 0, 0)$), and finally rescale all the coordinates by a factor R, we find that the hypersurface defined by

[8] Actually, this is how we originally found the solution

the equation $r = \kappa y$ is transformed to

$$(r_1^2 + r_2^2 + y^2 - R^2)^2 + 4R^2 r_2^2 = 4\kappa^2 R^2 y^2 \,. \tag{3.30}$$

Indeed, at the boundary of AdS, $y = 0$, this hypersurface reduces to the circle with $r_1 = R$, $r_2 = 0$. Writing this equation using the coordinates in (3.3) gives the solution found above: $\sin\eta = \kappa^{-1}\sinh\rho$.

Another way to find these hyper-surfaces is to notice that they are the orbits of the $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)$ subgroup of the four dimensional conformal group preserved by the circular loop [15]. Clearly this symmetry acts naturally on the $AdS_2 \times S^2$ surfaces we found.

We can also apply the conformal transformation to the four-form potential (this is most easily done in cartesian coordinates), obtaining

$$C_4' = \frac{L^4 r_2}{y^4}(r_1 dr_1 + y dy)\wedge d\psi \wedge dr_2 \wedge d\phi = \frac{L^4 \sinh^2\rho \sin\theta}{\sin^4\eta} d\rho \wedge d\psi \wedge d\theta \wedge d\phi \,. \tag{3.31}$$

The crucial point is that the conformal transformation yields a potential that differs from the one we used, by a component along the y direction. This comes about because the AdS isometries mix y with the rest of the coordinates.

The two potentials are related by a gauge transformation, $C_4' = C_4 + d\Lambda_3$, with $\Lambda_3 = -\frac{L^4 r_2}{2y^2} d\psi \wedge dr_2 \wedge d\phi$. It is important to note that were we to compute the WZ term with this C_4', we would just get the same answer as in the DBI term (that is only the first term in (3.9)). This would make the bulk action zero, as in the case of the straight line.

This is not surprising. Using C_4' just amounts to doing the calculation of the straight line in a different coordinate system. The subtle difference between the line and the circle is related here to the choice of four-form potential. One could find a similar ambiguity in the string calculation of the Wilson loop if one would regularize the divergence differently and remove the wrong boundary term. Likewise, in the perturbative calculation, the conformal transformation from the line to the circle amounts to a singular gauge transformation that adds a finite piece to the propagator [18]. That is a very close analog to the gauge transformation generated by Λ_3 above.

4. 't Hooft loop

The D3-brane calculation and the matrix model agreed for all finite values of the parameter κ at large N (so both $N \gg 1$ and $\kappa N \gg 1$), which in turn implies that $\lambda \gg 1$. So far the AdS calculation was valid only in the range $1 \ll \lambda \ll N$, where the string coupling is weak. We wish now to consider the case when $\lambda \gg N$, or strong coupling. To study it we will have to go to the S-dual theory, which is weakly coupled.

Under S-duality we replace g_s with $1/\tilde{g}_s$, or $\lambda = 16\pi^2 N^2/\tilde{\lambda}$. We can express κ in terms of the dual couplings as $\pi k/\sqrt{\tilde{\lambda}}$. We have to distinguish between two cases, when $1 \ll \tilde{\lambda} \ll N$, or $N \ll \lambda \ll N^2$, we should perform the calculation in the S-dual AdS space. When $\tilde{\lambda} \ll 1$, or $\lambda \gg N^2$, this AdS is highly curved, and instead we should use the dual gauge theory, which is now weakly coupled.

JHEP02(2005)010

S-duality does not only change the coupling, but also the charges. Electric charge is replaced with magnetic, so the Wilson loop we are studying will be an 't Hooft loop [24] in the dual theory.

4.1 't Hooft loop in AdS

The standard prescription for calculating an 't Hooft loop in the AdS/CFT correspondence is, like for the Wilson loop, by use of a minimal surface with the substitution of the fundamental string by a D1-brane. Again we will look to replace this D1-brane by a D3-brane, which happens very naturally in the presence of background flux, and goes under the name of the Myers effect [25]. We do the calculation here for a circular loop, the straight line can be done in a similar fasion.

The construction will be the same as before, the only difference is the replacement of the electric field by its Hodge dual, a magnetic field on the S^2 of our favorite coordinate system (3.3). Explicitly we shall take $F_{\theta\phi} = (k \sin\theta)/2$. Defined this way, k is the number of D1-branes immersed in the D3-branes.

The WZ part of the action is identical to the electric case (3.9), while the DBI part is now

$$\begin{aligned} S_{\mathrm{DBI}} &= T_{\mathrm{D3}} \int e^{-\Phi} \sqrt{\det(g + 2\pi\alpha' F)} \\ &= 2N \int d\rho \, d\theta \, \frac{\sin\theta \sinh^2\rho}{\sin^4\eta} \sqrt{\cos^2\eta (1+\eta'^2)\left(1 + \frac{\pi^2 k^2 \sin^4\eta}{\tilde{\lambda}\sinh^4\rho}\right)} \,. \end{aligned} \tag{4.1}$$

The solution to the equations of motion is, as before, $\sin\eta = \kappa^{-1}\sinh\rho$, now with $\kappa = \pi k/\sqrt{\tilde{\lambda}}$. On shell this part of the action is evaluated to be

$$S_{\mathrm{DBI}} = 4N(\kappa^2 + \kappa^4) \int d\rho \, \frac{1}{\sinh^2\rho} = -4N(\kappa^2+\kappa^4)\coth\rho \Big|_{\sinh\rho = \kappa\sin\eta_0}^{\sinh\rho=\kappa} \,. \tag{4.2}$$

Again we should replace the coordinate η with the conjugate momentum p_η by adding the boundary term

$$S_{\mathrm{boundary}} = -\eta_0 \int d\psi \, p_\eta = -4N\frac{\kappa}{\eta_0} \,. \tag{4.3}$$

This cancels one of the divergent terms in the DBI action. The other divergence, $4N\kappa^3/\eta_0$ will cancel against the divergent term in the WZ action.

Combining with the WZ term we find the final answer with the same functional form as before (3.16)

$$S_{\mathrm{circle}} = -2N\left[\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa\right] . \tag{4.4}$$

There are some important differences from the electric case. While there we had to replace the field strength by its dual, for the case at hand the magnetic field is the correct variable counting the number of D1-branes dissolved in the D3-brane. Consequently the bulk action had the exact same linear divergence found when calculating the 't Hooft loop by means of D1-branes.

4.2 't Hooft loop in perturbation theory

In the regime when $\lambda \gg N^2$, or $\tilde{\lambda} \ll 1$ we can no longer use the dual AdS, and instead have to study the weakly coupled dual gauge theory. The expectation value of an 't Hooft loop in four dimensions was never calculated, as there are some technical hurdles that are yet to be overcome. Still we will try to carry this calculation as far we can.[9]

We may try to extend the results of the matrix model for finite N to this regime. In that case we need to evaluate the Laguerre polynomial in (3.22) at very large negative arguments, where it will be dominated by its highest exponent $L_n^k(x) \sim 1/n!(-x)^n$. This gives

$$\langle V \rangle = \frac{1}{N} L^1_{N-1}(-4N\kappa^2) \exp[2N\kappa^2] \sim \frac{1}{N!} \left(\frac{2\pi k}{\tilde{g}_{YM}} \right)^{2N-2} \exp\left[\frac{2\pi^2 k^2}{\tilde{g}^2_{YM}} \right]. \tag{4.5}$$

A very similar result is found from the Wilson loop and 't Hooft loop calculation in terms of the D3-brane. The result (3.16) expanded for large κ is

$$S = -2N\left(\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa \right) \sim -2N\left(\kappa^2 + \ln 2\kappa + \frac{1}{2} + \cdots \right). \tag{4.6}$$

Switching again to the dual variables we find that the 't Hooft loop is given by

$$\langle V \rangle = \exp[-S] \sim \left(\frac{e}{N} \right)^N \left(\frac{2\pi k}{\tilde{g}_{YM}} \right)^{2N} \exp\left[\frac{2\pi^2 k^2}{\tilde{g}^2_{YM}} \right]. \tag{4.7}$$

Unlike the Wilson loop calculation, where at large N the $\sqrt{\lambda}$ behavior appeared, there seems to be no subtlety in taking the large N limit, on the 't Hooft loop. The two results agree.

It is not too hard to explain the different factors in this expression in terms of a perturbative gauge theory calculation for $k = 1$ [26].

The 't Hooft loop is a topological defect creating a magnetic source in the non-abelian gauge theory. For the circular source it is useful to employ the coordinates ρ, ψ, θ and ϕ of (3.2) with $\eta = 0$, so the flat space metric is

$$ds^2 = \frac{R^2}{(\coth\rho - \cos\theta)^2} \left[\frac{1}{\sinh^2\rho}(d\rho^2 + d\psi^2) + d\theta^2 + \sin^2\theta d\phi^2 \right]. \tag{4.8}$$

In these coordinates the source is at $\rho = 0$.

The expectation value of the 't Hooft loop should be given by the partition function in the background generated by this magnetic source (which in the supersymmetric case also carries scalar charge of the field we label X^9). At the classical level this magnetic field will sit in one U(1) factor, and we can write the form of the electromagnetic field configuration. It is given by

$$\begin{aligned} X^9 &= \frac{1}{2R}(\coth\rho - \cos\theta)\,, \\ A_\phi &= \frac{1}{2}(\pm 1 - \cos\theta)\,, \end{aligned} \tag{4.9}$$

[9] Based in part on a collaboration between N.D. and N. Itzhaki [26].

where the sign choice in the gauge field corresponds to the two gauges covering the north or south pole of the sphere. The straight forward way to get this expression is by solving the magnetostatic problem for a monopole source along the circle (as well as the more familiar scalar). The Laplacean in our coordinate system is closely related to that on $AdS_2 \times S^2$, and is easy to invert. Another way is to start with the straight line and do the conformal transformation to the circle. Finally one can just notice that this is the same solution as that of the DBI action with X^9 replacing $1/y$.

The classical action for this configuration includes two terms, from the gauge field and from the scalar. Both are divergent, but it is easy to regularize the integrals and extract a finite answer

$$S = \frac{1}{4\tilde{g}_{YM}^2} \int \sqrt{g}\left(F^2 + (\partial X^9)^2\right) \sim -\frac{2\pi^2}{\tilde{g}_{YM}^2} . \tag{4.10}$$

While at the technical level it is easy to subtract the divergence, it should really not exist in the first place. In all the other calculations the complete final result, including perhaps boundary terms is finite. This is true for the calculations of the Wilson loop as well as the 't Hoof loop in AdS using either fundamental and D-strings or D3-branes. The same is true for the perturbative calculation of the Wilson loop. This is an indication that we are missing some terms in the action localized on the defect. At the classical level they will merely fix this divergence, hence we are able to continue without fully understanding them.

The leading behavior of the 't Hooft loop will be the exponent of minus this classical action, and indeed we see that this result agrees with the matrix model and D3-brane calculations. At the semi-classical level one has to quantize the theory around this background and evaluate the fluctuation determinant of all the fields. This calculation is naturally broken up into the zero mode contribution and that of the massive fluctuations. The calculation of the latter is beyond the scope of the present paper, and we will only comment on the zero-mode determinant.

The classical solution was an abelian ansatz living in a U(1) factor within U(N), breaking the gauge symmetry to U($N-1$) $\times$ U(1). This breaking results in $2N-2$ zero modes each contributing a factor of $1/\tilde{g}_{YM}$ to the one-loop determinant. More precicely, those zero modes parameterize a coset manifold and their contribution is just the volume of this coset.

The volume of this manifold can be calculated in similar ways to that done for the zero modes of instantons (see for example [27]). If one normalizes the generators of U(N) such that $\mathrm{Tr}T^aT^b = \delta^{ab}/2$, the volume of U($N$) can be written in terms of the volume of U($N-1$) and that of the $2N-1$ dimensional unit sphere $(2\pi^N/(N-1)!)$ as

$$V(\mathrm{U}(N)) = 2^{2N-3/2} V(S^{2N-1}) V(\mathrm{U}(N-1)) . \tag{4.11}$$

The subgroup that is preserved by our ansatz is the product of U($N-1$) and a U(1) of radius $2\pi\sqrt{2}$. The result one gets is

$$\frac{(4\pi)^{N-1}}{(N-1)!\,\tilde{g}_{YM}^{2N-2}} . \tag{4.12}$$

JHEP02(2005)010

Together the classical action and the zero modes give the answer

$$\langle V\rangle \sim \frac{1}{(N-1)!}\left(\frac{2\sqrt{\pi}}{\tilde{g}_{YM}}\right)^{2N-2}\exp\left[\frac{2\pi^2}{\tilde{g}_{YM}^2}\right]. \tag{4.13}$$

This result is remarkably close to (4.5). The discrepancy in the powers of π may be fixed by the determinant of the massive fluctuations. The source of the missing power of N is unclear and could be related to the normalization of the 't Hooft operator, if it is dual to the Wilson loop defined without the factor of $1/N$ before the trace.

We should note that the calculations in the entire paper are concerned with $\mathrm{U}(N)$ gauge theory. Everything can be generalized to $\mathrm{SU}(N)$ in a reasonably simple manner. This is of importance for the 't Hooft loop, since it is a well defined operator only in the latter case. The resulting modifications are the rescaling of the action by a factor of $(N-1)/N$ and the decrease in the zero mode determinant by a factor of $N-1$.

While this perturbative calculation captured some of the salient features of the 't Hooft loop expectation value as evaluated from the matrix model and AdS, it suffers from some serious flaws. To do any better one may have to add to the gauge theory new degrees of freedom living along the loop, giving a defect CFT, like in [28].

5. Discussion

We have found D3-brane solutions in AdS_5 that carry electric flux and end along a curve on the boundary. This is the correct prescription for calculating the expectation value of a Wilson loop multiply wrapped around that curve when the multiplicity of the loop k is big.

The expectation value of the Wilson loop calculated this way includes sereval different parts, the bulk contribution comprising of the DBI and the WZ pieces, and in addition the boundary terms were crucial to finding the right answer. One has to replace the coordinate transverse to the AdS boundary with its conjugate momentum, and apply the same procedure to the gauge field (in the electric case).

Each of the different terms in the action had a linear divergence, but they all canceled in the final answer. In the case of the straight line the final result was zero, while for the circle there was a finite remnant.

In the circular example, where the answer is nontrivial, we found the D3-brane solution to reproduce correctly the string result $k\sqrt{\lambda}$ as well as an infinite series of corrections. The first of those $k^3\lambda^{3/2}/96N^2$ is seen as the first string-loop correction to the above result. The full D3-brane solution includes corrections to all orders in $1/N$ that are leading at large λ.

Thus we find that

> *The action of the D3-brane carrying electric flux and ending along the loop on the boundary captures correctly the action of the analogous fundamental strings as well as the entropy of summing over semi-classical string surfaces of <u>all</u> genus.*

Furthermore in this case we were able to compare the results to the gaussian matrix model and found perfect agreement.

It is amusing to compare the k dependence we found for this Wilson loop in a conformal theory, and the tension of k-strings in supersymmetric confining theories, as given by the Douglas-Shenker formula

$$\sigma_k = N\Lambda^2 \sin\frac{\pi k}{N} = \Lambda^2 \pi k - \frac{\Lambda^2\pi^3}{3!}\frac{k^3}{N^2} + \cdots . \tag{5.1}$$

In both cases the leading term scales with k, and more interestingly, the first correction in a large N expansion goes like k^3/N^2. This k^3 scaling is not expected a priori, and it would be nice to develop a intuitive understanding of it (see [29] for a heuristic picture of this scaling for k-strings).

It would be very interesting to study other Wilson loops using this prescription. Probably the most interesting example would be the pair of anti-parallel lines. The standard string result for a pair of k coincident lines at a distance r gives the effective potential $V(r) = -4k\pi^2\sqrt{2\lambda}/(\Gamma(1/4)^4 r)$. Finding the D3-brane would allow us to deduce the corrections to the effective multi-string tension beyond this leading behavior. It is natural to expect that the correction will again be of order $k^3\lambda^{3/2}/N^2$, but one would have to do the full calculation to find the numerical coefficient.

We expect the general features of the D3-brane solution to carry over to all other Wilson loops including the parallel lines. In the UV any smooth loop looks like a straight line, thus the D3-brane will have the same asymptotic form, with the geometry approaching $AdS_2 \times S^2$. All the divergences come from the UV and will cancel as in the two examples studied here, but a finite contribution, dependent on the shape of the curve, will remain.

Let us note that while we got those non-planar contributions to the Wilson loop quite easily for the circle by using the D3-branes, there should be other ways of calculating them. One direction is to follow [14] and calculate the exchange of supergravity fields between different parts of the string surface. Another approach would be to follow the argument given in [18] and reviewed above for the leading power of λ in the first $1/N^2$ correction. The factor of 1/96 should come out of the ratio of volumes of the moduli spaces of degenerate genus one Riemann surfaces divided by the genus zero case.

Another interesting direction is to look at other terms in the D3-brane action. There are curvature corrections, which would correspond to terms that are subleading in λ to the ones we found (or subleading in N, if we keep κ constant). Those terms can also be calculated from the string solution by using world-sheet techniques, but while the prescription for performing calculation is straight-forward, the calculation itself is pretty hard [30]. It may, therefore, be advantageous to use the D3-brane for this purpose, and again compare the result to the matrix model.

Other terms in the action correspond to D-instanton contributions. Those should agree with the instanton correction to the perturbative result [15] (which are not captured by the matrix model).

From the supergravity perspective the difference between a multiply wrapped Wilson loop and an operator with more than one trace is subtle. They are all described by k coincident fundamental strings, where the difference is whether the strings are independent, or connected along branch points to a single Riemann surface [9]. From the comparison to the matrix model it seems like the single D3-brane solution corresponds to the single trance operator in the gauge theory. It's natural to guess that in the general case we should have one D3-brane for each trace in the Wilson loop operator (or each disconnected string surface). That system may then be an excellent laboratory for studying the non-abelian generalization of the DBI action.

It is quite remarkable how this gaussian matrix model [18] captures interesting string phenomena. It is successful in reproducing the AdS calculation of the Wilson loop using classical strings and all the higher genus corrections to it, or the D3-brane solution. The matrix model still includes many more terms on top of the ones studied here, and a few more of them will be compared to AdS in [31].

Acknowledgments

We would like to thank Ofer Aharony, Jan Ambjørn, Tom Banks, Micha Berkooz, Sunny Itzhaki, Yuri Makeenko, David Mateos, John McGreevy, Rob Myers, Joan Simón and Kostas Skenderis for interesting conversations. We are also grateful to Soo-Jong Rey for pointing out the special relevance of [2]. N.D. would like to thank the Weizmann Institute, where this collaboration was initiated, for its hospitality as well as the Aspen Center for Physics and the University of Amsterdam. B.F. is grateful for the hospitality of the Aspen Center for Physics and the University of California at Santa Cruz.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S.J. Rey and J.T. Yee, *Macroscopic strings as heavy quarks in large-N gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J.* **C 22** (2001) 379 [hep-th/9803001].

[3] J.M. Maldacena, *Wilson loops in large-N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [hep-th/9803002].

[4] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from anti-de Sitter space*, *J. High Energy Phys.* **06** (2000) 008 [hep-th/0003075].

[5] M.T. Grisaru, R.C. Myers and O. Tafjord, *SUSY and Goliath*, *J. High Energy Phys.* **08** (2000) 040 [hep-th/0008015].

[6] A. Hashimoto, S. Hirano and N. Itzhaki, *Large branes in AdS and their field theory dual*, *J. High Energy Phys.* **08** (2000) 051 [hep-th/0008016].

[7] C.G. Callan Jr. and J.M. Maldacena, *Brane dynamics from the Born-Infeld action*, *Nucl. Phys.* **B 513** (1998) 198 [hep-th/9708147].

[8] G.W. Gibbons, *Born-Infeld particles and Dirichlet p-branes*, *Nucl. Phys.* **B 514** (1998) 603 [hep-th/9709027].

[9] D.J. Gross and H. Ooguri, *Aspects of large-N gauge theory dynamics as seen by string theory*, *Phys. Rev.* **D 58** (1998) 106002 [hep-th/9805129].

[10] M.R. Douglas and S.H. Shenker, *Dynamics of* $\mathrm{SU}(N)$ *supersymmetric gauge theory*, *Nucl. Phys.* **B 447** (1995) 271 [hep-th/9503163].

[11] A. Hanany, M.J. Strassler and A. Zaffaroni, *Confinement and strings in M QCD*, *Nucl. Phys.* **B 513** (1998) 87 [hep-th/9707244].

[12] C.P. Herzog and I.R. Klebanov, *On string tensions in supersymmetric* $\mathrm{SU}(M)$ *gauge theory*, *Phys. Lett.* **B 526** (2002) 388 [hep-th/0111078].

[13] N. Drukker, D.J. Gross and H. Ooguri, *Wilson loops and minimal surfaces*, *Phys. Rev.* **D 60** (1999) 125006 [hep-th/9904191].

[14] D. Berenstein, R. Corrado, W. Fischler and J.M. Maldacena, *The operator product expansion for wilson loops and surfaces in the large-*N *limit*, *Phys. Rev.* **D 59** (1999) 105023 [hep-th/9809188].

[15] M. Bianchi, M.B. Green and S. Kovacs, *Instanton corrections to circular wilson loops in* $N = 4$ *supersymmetric Yang-Mills*, *J. High Energy Phys.* **04** (2002) 040 [hep-th/0202003].

[16] A. Mikhailov, *Special contact Wilson loops*, hep-th/0211229.

[17] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in* $N = 4$ *supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 582** (2000) 155 [hep-th/0003055].

[18] N. Drukker and D.J. Gross, *An exact prediction of* $N = 4$ *SUSY M-theory for string theory*, *J. Math. Phys.* **42** (2001) 2896 [hep-th/0010274].

[19] J.P. Gauntlett, C. Koehl, D. Mateos, P.K. Townsend and M. Zamaklar, *Finite energy Dirac-Born-Infeld monopoles and string junctions*, *Phys. Rev.* **D 60** (1999) 045004 [hep-th/9903156].

[20] C. Bachas and M. Petropoulos, *Anti-de-Sitter D-branes*, *J. High Energy Phys.* **02** (2001) 025 [hep-th/0012234].

[21] K. Skenderis and M. Taylor, *Branes in AdS and pp-wave spacetimes*, *J. High Energy Phys.* **06** (2002) 025 [hep-th/0204054].

[22] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[23] R. Emparan and H.S. Reall, *A rotating black ring in five dimensions*, *Phys. Rev. Lett.* **88** (2002) 101101 [hep-th/0110260].

[24] G. 't Hooft, *On the phase transition towards permanent quark confinement*, *Nucl. Phys.* **B 138** (1978) 1.

[25] R.C. Myers, *Dielectric-branes*, *J. High Energy Phys.* **12** (1999) 022 [hep-th/9910053].

[26] N. Drukker and N. Itzhaki, unpublished.

[27] C.W. Bernard, *Gauge zero modes, instanton determinants and quantum-chromodynamic calculations*, *Phys. Rev.* **D 19** (1979) 3013.

[28] O. DeWolfe, D.Z. Freedman and H. Ooguri, *Holography and defect conformal field theories*, *Phys. Rev.* **D 66** (2002) 025009 [hep-th/0111135].

[29] A. Armoni and M. Shifman, *Remarks on stable and quasi-stable K-strings at large-*N, *Nucl. Phys.* **B 671** (2003) 67 [hep-th/0307020].

[30] N. Drukker, D.J. Gross and A.A. Tseytlin, *Green-Schwarz string in $AdS_5 \times S^5$: semiclassical partition function*, *J. High Energy Phys.* **04** (2000) 021 [hep-th/0001204].

[31] N. Drukker and B. Fiol, in preparation.